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A Study of Markov Branching Process with Varying Growth Rate

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Abstract **- Branching processes in random environments occurs when the values of the reproduction probabilities and instantaneous reproduction rates are themselves sample paths of stochastic processes. These processes occur very naturally as models to describe the growth mechanism of biological systems. In this paper, we consider a Markov branching process with varying growth rates controlled by a random environment described by an alternating renewal process; we will derive expression for the mean number of the size of the population.**
Keywords: **Markov Branching Process, Varying growth rate, Moment generating function.**

I. INTRODUCTION

The parameters of the branching processes are allowed to be functions of time only. Klebaner and Schuh (1982) obtained a connection between the limit and the maximum random variable of branching processes in varying environments. [3]MacPhee and Schuh(1983) have analyzed Galton-Watson branching process in varying environments with essentially constant mean and two rates of growth. Biggins and D'Souza (1982) and Souza and Biggins (1992) considered supercritical Galton-Watson process in varying environments and obtained some limit theorems.
In the study of branching processes in random and varying environments[5], the stochastic integral

$$Y(t)=\int_0^t Z(u)du \qquad 1.1$$

plays a dominant role by representing a quantity associated with response phenomena of the underlying branching process Z(t). The integral (1.1) exits almost surely for almost all branching processes and its biological important.

II. THE BRANCHING PROCESS IN RANDOM ENVIRONMENT:

Consider a branching process Z(t) initiated by one particle at time t = 0. Assume that the life-Time of each particle is exponentially distributed with parameter $\lambda > 0$ and that the process evolves as a Markov branching process $Z_2(t)$. Then

$$Z(t)=\begin{cases} Z_1(t) 0\le t\le T \\ Z_2(t)\ t>T \end{cases} \qquad 2.1$$

Let F(t) be the distribution function of T and $F'(t)=f(t)$ .Also , let $h_j(s)$,j=1,2,be the off-spring probability generating functions of the Markov branching processes. $Z_j(t)$, j = 1,2, respectively. Consider the stochastic integrals.

$$Y(t)=\int_0^t Z(u)\,du \qquad 2.2$$

$$Y_1(t)=\int_0^t Z_1(u)\,du \qquad 2.3$$

$$Y_2(t)=\int_0^t Z_2(u)du \qquad 2.4$$

The Stochastic integrals (2.2) (2.3) & (2.4) exist almost surely probability space. Let us define the joint-moment generating functions.

$$G(s_1,s_2,t)=E\{s_1^{Z(t)}e^{-s_2Y(t)}\,|Z(0)=1\},$$

$$G_j(s_1,s_2,t)=E\{s_1^{Z_j(t)}e^{-s_2Y_j(t)}\,|Z_j(0)=1\},\ \ j=1,2.$$

Puri (1969) has shown that

$$G_j(s_1,s_2,t)=s_1e^{-(s_2+\lambda)t}+\lambda\int_0^t e^{-(s_2+\lambda)u}\,h_j\,(G_j(s_1,s_2,t-u))\,du,\ j=1,2. \qquad 2.5$$

Now using the regenerative structure of the process Z(t), the following integral equation can be derived.

$$G(s_1,s_2,t)=\{1-F(t)\}\,G_1(s_1,s_2,t)\ +\int_0^t f(u)\,G_1(1,s_2,u)\ G_1(G_2(s_1,s_2,t-u)\,0,u)du, \qquad 2.6$$

III. RANDOM ENVIRONMENT AND JOINT-MOMENT GENERATING FUNCTION

For obtaining explicit expressions for the means of Z(t) and Y(t), let us assume that $f(t) = \mu e^{-\mu t}, t>0, \mu>0.$
Then the equation (2.6) becomes.

$$G(s_1,s_2,t)=e^{-\mu t}G_1(s_1,s_2,t)+\mu\int_0^t e^{-\mu u}G_1(1,s_2,t)\,G_1(G_2(s_1,s_2,t-u)0,u)du, \qquad 3.1$$

Differentiating (3.1) with respect to s_1 at ($s_1= 1, s_2 = 0$) , we obtain

$$m_z(t)=e^{-\mu t}\,mz_1(t)+\mu\int_0^t e^{-\mu u}\,mz_1(u)\,mz_2\,(t-u)\,du. \qquad 3.2$$

Where we have put

$$m_Z(t)=E\{Z(t)|Z(0)=1\}$$

$$m_{Z_j}(t)=E\{Z_j(t)|Z_j(0)=1\},\, j=1,2.$$

Since $Z_j(t)$ is markov branching, We get

$$m_{Z_j}(t)=e^{\alpha jt}\,,\, j=1,2.$$

Where

$$\alpha_j=\lambda(h_j^{'}(1)-1),\, j=1,2.$$

Consequently, the equation (3.2) gives

$$m_Z(t)=\frac{(\alpha_1-\alpha_2)\,e^{(\alpha_1-\mu)t}-\mu e^{\alpha_{2t}}}{\alpha_1-\alpha_2-\mu} \qquad 3.3$$

On other hand, if we differentiate (3.1) with respect to s_2 at ($s_1= 1, s_2 = 0$), we obtain

$$m_Y(t)=e^{-\mu t}m_{Y_1}(t)+\mu\int_0^t e^{-\mu u}\left\{m_{Y_1}(u)+\, m_{Z_1}(u)\,m_{Y_2}(t-u)du\right. \qquad 3.4$$

Where we have written

$$m_Y(t)=E[Y(t)], \qquad m_{Y_1}(t)=E[Y_j(t)],\, j=1,2.$$

From Puri (1969), we have

$$m_{Y_1}(t)=\frac{e^{\alpha_i t}-1}{\alpha_i},\alpha_i\neq 0, i=1,2.$$

And hence the equation (3.4) gives

$$m_Y(t)=e^{-\mu t}\,\{\frac{e^{\alpha_i t}-1}{\alpha_1}\}+\frac{\mu}{\alpha_1}\{\frac{e^{(\alpha_1-\mu)t}-1}{\alpha_1-\mu}\}+\{\frac{e^{-\mu t}-1}{\mu}\}+\frac{\mu}{\alpha_2}\{\frac{e^{(\alpha_1-\mu)t}-e^{\alpha_{2t}}}{\alpha_1-\alpha_2-\mu}-\frac{e^{(\alpha_1-\mu)t}-1}{\alpha_1-\mu}\}.$$

IV. THE HIGHER ORDER CHARACTERISTICS OF BRANCHING PROCESS

The second factorial moment of Z(t) is defined by

$$m_Z^{(2)}(t) = E\{Z(t)(Z(t)-1) \mid Z(0)=1\}$$

To get an expression for $m_Z^{(2)}(t)$, we differentiate (3.1) twice with respect to s_1 at (s_1= 1, s_2 = 0). We get.

It will know Puri (1969) that

$$m_{Z_j}^{(2)}(t) = (A_j+1)(e^{2\alpha_j t} - e^{\alpha_j t}),\ j=1,2.$$

Where

$$A_j = \frac{h_j^{''} - h_j^{'}(1)+1}{h_j^{'}(1)-1}, \alpha_j \neq 0,\ j=1,2.$$

V.THE CORRELATION BETWEEN BRANCHING PROCESS AND MGF :

For the purpose of illustration, we assume that both $Z_1(t)$, and $Z_2(t)$ are of binary splitting (i.e., $h_j(s) = p_j + (1-p_j)s^2$, $j=1,2.$) Then we have $h_j^{'}(1) = h_j^{''}(1)$ so that $A_j = \frac{1}{h_j^{'}(1)-1}$, $j=1,2.$ We consider two non-critical cases

Case(i)

We assume the following values for the parameter.

$$\lambda = 1.25,\ \mu = 0.5\ ,\ h_j^{'}(1) = 0.5\ ,\ h_2^{'}(1) = 1.5$$

We compute the values for $m_Z(t), m_Y(3), Var(z(t)), Var(Y(t))$.

And the correlation coefficient ρ between Z(t) and Y(t) and present them in table (1). From the table we find that the correlation coefficient increases as time increases.

Case(ii)

We assume the following values for the parameters

$$\lambda = 1.25\ ,\ \mu = 0.5\ ,\ h_j^{'}(1) = 0.5\ ,\ h_2^{'}(1) = 0.5$$

TABLE I CORRELATION COEFFICIENT P BETWEEN Z(T) AND Y(T)

t	E(Z(t))	E(Y(t))	VAR(Z(t))	VAR(Y(t))	ρ
0.2	0.89	0.19	0.21	0.00	0.84
0.4	0.82	0.36	0.39	0.02	0.82
0.6	0.78	0.52	0.58	0.07	0.81
0.8	0.76	0.67	0.81	0.15	0.81
1.0	0.77	0.83	1.12	0.30	0.82
1.2	0.79	0.98	1.54	0.55	0.83
1.4	0.83	1.14	2.11	0.93	0.85
1.6	0.89	1.32	2.87	1.52	0.87
1.8	0.97	1.50	3.89	2.42	0.88
2.0	1.07	1.71	5.24	3.74	0.90
2.2	1.19	1.93	7.02	5.67	0.91
2.4	1.33	2.18	9.34	8.43	0.92
2.6	1.49	2.47	12.38	12.34	0.93
2.8	1.67	2.78	16.34	17.82	0.94
3.0	1.89	3.14	21.49	25.42	0.95

TABLE II CORRELATION COEFFICIENT P BETWEEN Z(T) AND Y(T)

t	E(Z(t))	E(Y(t))	VAR(Z(t))	VAR(Y(t))	ρ
0.2	1.12	0.21	0.30	0.00	0.86
0.4	1.23	0.45	0.70	0.03	0.85
0.6	1.34	0.70	1.22	0.12	0.85
0.8	1.44	0.98	1.87	0.31	0.84
1.0	1.53	1.28	2.68	0.65	0.84
1.2	1.62	1.59	3.66	1.22	0.84
1.4	1.71	1.93	4.84	2.10	0.84
1.6	1.79	2.28	6.26	3.39	0.83
1.8	1.87	2.64	7.95	5.22	0.83
2.0	1.95	3.03	9.96	7.74	0.83
2.2	2.03	3.42	12.34	11.11	0.83
2.4	2.10	3.84	15.15	15.57	0.83
2.6	2.18	4.26	18.45	21.35	0.82
2.8	2.25	4.71	22.34	28.77	0.82
3.0	2.32	5.16	26.91	38.16	0.82

VI. CONCLUSION

We compute the values for $m_Z(t), m_Y(t), Var(z(t)), Var(Y(t))$ and the correlation coefficient ρ between Z(t) and Y(t) and we can prove that the correlation coefficient decreases as time increases. Numerically we can give examples.

REFERENCES

[1] Assmusssen,S. and Hering, H.(1983): *Branching Processes*. Birkhauser, Boston.

[2] Athreya,K.B. and Kaplan, N. (1976) : *limit heorems for a branching process with disaster*. J.Appl.Probab.13, 466-475.

[3] Athreya,K.B. and Karlin, N. (1976a) : *Branching processes with random environments.* Bull.Amer.Math.Soc.76, 865-870.

[4] Athreya,K.B. and Karlin, N (1971b) : *Branching processes with random environments I.* Extinction probabilities. Ann Math. Statist. 42, 1499-1520.

[5] Athreya,K.B. and Karlin, N (1971c) : *Branching processes with random environments II.* Limit Theorems. Ann Math. Statist. 42, 1843-1858.

[6] Athreya,K.B. and Ney, P.E.(1972): *Branching Processes*. Spring-Verlag, Berlin.

[7] Bellman, J.D. and Harris, T.E. (1948): *On the theory of age-dependent Stochastic Branching processes.* Proc. Nat. Acad. Sci., N.Y., 34, 601-604.

[8] Bharucha-Reid, A.T. (1960): *Elements of the theory of Markov Processes and their Applications.* McGraw-Hill Book Company, New York.

[9] Harring, D.P.and Fleming, T.R. (1978): Estimation for branching processes with varying and random environment, math. Biosci., 39, 255-271.

[10] Harris, T.E. (1963): The Theory of branching Processes.Berlin: Springer-Verlag.

[11] Kaplan, N. (1973): *A continuous time markov Branching model with random environments,* Adv. Appl. probab. 5, 37-54

[12] Macphee, I.M. and Schuh, H.J. (1983): *a Galton-Watson branching process in varying environments with essentially constant means and rates of growth.* Austal. J. Staist. 25, 329-338.

[13] Parthasarathy, R.(1979); *On modified Markov branching process*. J. Math, Biol. 7,95-97.

[14] Parthasarathy, R.. and krisnamoorthy, A. (1979); *Markov branching process with varying growth rates*. Bull. Math. Biol. 41, 607-610.

[15] Smith, W.L. and Wilkinson, W. (1969): *On branching processes in random environments.* Ann. Math. Statist. 40. 814-827.

[16] Tanny, D. (1977): *Limit Theorems for branching processes in a random environment.* Anna. Probab. 5, 100-116.