Asian Journal of Science and Applied Technology
ISSN: 2249-0698 (P) Vol.5 No.2, 2016, pp.10-14

DOI: https://doi.org/10.51983/ajsat-2016.5.2.934

A Study on Queueing Systems Associated Random Evolutions With Branching Process

M.Reni Sagayaraj[1], S.Anand Gnana Selvam[2*] and R.Reynald Susainathan[3]
[1,2]Department of Mathematics, Sacred Heart College(Autonomous),
Tirupattur - 635601, Vellore District, Tamil Nadu, S.India.
[3]Reckitt Benckiser, India Limited (Gurgaon), New Delhi. India.
E-mail: anandjuslin@gmail.com
(Received 20 June 2016; Revised 14 July 2016; Accepted 3 August 2016; Available online 10 August 2016)

Abstract **- In this paper we deal with branching process in which each particle procedure offspring at the time of completion of its life-time L according to a probability generating functions. We obtain an integral equation for the joint generating function of the vector process and limiting distribution.**
Keywords: **Markov branching processes, joint generating function, restricted moment, performance analysis**

I.INTRODUCTION

Branching processes occur in the study of several physical and biological phenomena where each particle remains for some amount of time and then splits into several particles of same type. In physical science they are called reproductive processes. Two important factors which characterize each of these processes are life time and the number of off-spring of each self-branching particle. By considering various assumptions on the life time, branching processes have been classified into Galton-Waston branching processes, Markov branching processes and age-dependent branching processes and have been studied very extensively in the past Harris (1963), Srinivasan (1969), Mode (1971), Athreya and Ney (1972) and Assmussen and Hering (1983). Recently, Pakes (2000) prepared a report on biological applications of branching processes, which is wider in scope (it has a lot spatial branching and ecology. In markov branching processes and age-dependent branching processes which come under the category of continuous-time branching processes, there is no restriction on the time of splitting of each particle. Accordingly, the study of continuous-time branching processes with the above restriction is absolutely essential. In the present paper, we investigate a branching process which incorporates restricted random evolutions with branching process. Accordingly, the study of continuous-time branching processes with above restriction is absolutely essential. In this paper, we investigate a branching process which incorporates age restriction for branching, and derived associated tom stochastic integral. The organization of the chapter is as follows. In section 2, the restricted branching process is described. the joint moment generating function of the branching process and an associated stochastic integral is obtained in section 3. In section 4, explicit expressions for the mean of the branching process and the stochastic integrals are found. In section 5 the limit distribution of Y(t) on the evolution of the stochastic integral is derived.

II. RESTRICTED BRANCHING PROCESSES

We start with one particle at time $t = 0$. The particle lives for a random length L of time and at the time of its death leaves a random number of identical off-springs provided $T_1 < L < T_2$. Where T_1 and T_2 are positive constants. The particle Leaves no descendants if $L < T_1$ or $L > T_2$. The descendants behave independently and indentically to the ancestor. Let $G(t)$ be the distribution function of the life-time and $h(s)$ be the off-spring probability generating function of each particle. Let $X(t)$ be the number of particles present at ime t. Then, we call the stochastic process $\{X(t), t \geq 0\}$ or simply $X(t)$ as restricted branching process. Firstly, if $T_1 = 0$ and $T_2 = \infty$, then the branching process $X(t)$becomes the Bellmen-Harris branching process also called age-dependent branching process, see (1963). Secondly, if $T_1 = 0$, $T_2 = \infty$, and $F(t) = 1 - e^{-\lambda t}$ $t \geq 0$,$\lambda > 0$ then $X(t)$ becomes the Markov branching process.

III. A STOCHASTIC INTEGRAL AND THE JOINT M.G.F

We consider the restricted Markov branching process$X(t)$, where $X(0) = 1$. Let ω be any element of the outcome space Ω of $X(t)$. Then, for all ω except in a set of measure zero, the sample path $X(t, \omega)$ is a step-function and hence $X(t, \omega)$ is integrable over any finite interval of the time-axis. Accordingly, the integral $\int_0^t X(u)du$ is called a stochastic integral exists almost surely and defines a random variable $Y(t)$ for each $t > 0$. The stochastic process $\{Y(t), t > 0\}$ or simply $Y(t)$ has been studied under different context in literature (see Puri(1966a,b,1969), Jagers (1967), Pakes (1975), Srinivasan and Udayabaskaran (1982), and Udayabaskaran and Sudalaiyandi(1986)).
We consider the vector process $\big(X(t), Y(t)\big)$ and define its joint-moment generating function by
$G(s_1, s_2, t) = E\left\{s_1^{X(t)} e^{-s_2 Y(t)} \mid X(0) = 1\right\}$
Considering the three probabilities $0 < t < T_1$, $T_1 < t < T_2$ and $t > T_2$, observing that the ancestor existing at time 0 ends its life either before or after t and using renewal type

arguments, we obtain the following integral equation for $G(s_1, s_2, t)$:

$$G(s_1,s_2,t) = s_1 e^{-(s_2+\lambda)t} + \lambda \int_0^t e^{-(s_2+\lambda)\tau}\, d\tau + \lambda \int_{T_1}^t e^{-(s_2+\lambda)\tau} h\big(G(s_1,s_2,t-\tau)\big)\, d\tau,\ T_1 < t < T_2 \quad (3.1)$$

Using $G(s_1, s_2, t)$, we can obtain the moment the structure of $X(t)$ *and* $Y(t)$. In the next section, we obtain explicit expressions for $E[X(t)]$ *and* $E[Y(t)]$.

IV.THE MEAN OF $X(t)$ and $Y(t)$

We use the following notation for the means:

$M_X(t) = E\{X(t)|X(0) = 1\}$,

$M_Y(t) = E\{Y(t)|Y(0) = 1\}$

We shall first obtain $M_X(t)$. For this, we differentiate (1) with respect to s_1 at $(s_1 = 1,\ s_2 = 0)$ and we get

$$M_X(t) = e^{-\lambda t} + \lambda m \int_{T_1}^t e^{-\lambda\tau} M_X(t-\tau)\, d\tau,\ T_1 < t < T_2 \quad (4.1)$$

Where we have put $m = h'(1)$**.** the constant m represents the expected number of progeny of each individual.

The equation (2.1) can be solved by adopting an iterative procedure as explained below:

From the equation (2.1), we have explicitly

$$M_X(t) = e^{-\lambda t}, 0 < t < T_1 \quad (4.2)$$

To obtain explicit solution for $M_X(t)$ in the other cases $T_1 < t < T_2$ and $t > T_2$, we assume $T_2 = \upsilon T_1$, where υ is a positive integer ≥ 2.

Case: (i). $T_1 < t \leq \upsilon T_1$

We divide the interval $(T_1, \upsilon T_1]$ into the subintervals $(T_1, 2T_1], \dots\dots, ((\upsilon-1)T_1, \upsilon T_1]$ and obtain $M_X(t)$ successively in each of them.

Sub case: $\boldsymbol{T_1 < t \leq 2T_1}$

Her, we note that $0 < t - T_1 \leq T_1$ and hence by (2.2), we have

$$\int_{T_1}^t e^{-\lambda\tau} M_X(t-\tau)\, d\tau = \int_0^{t-T_1} e^{-\lambda(t-\tau)} M_X(\tau) d\tau = \int_0^{t-T_1} e^{-\lambda(t-\tau) e^{-\lambda\upsilon}}\, d\upsilon \quad (4.3)$$

Sub case: $\boldsymbol{l = \nu}$

Here, we clearly have $\nu T_1 < t \leq (\nu+1)T_1$ so that $(\nu-1)T_1 < t - T_1 \leq \nu T_1$. We split the interval $(T_1, \upsilon T_1]$ into subintervals

$$(T_1, t-(\nu-1)T_1], (t-(\nu-1)\ T_1, t-(\nu-2)\ T_1], \dots, (t-2\ T_1, t-\ T_1], (t-\ T_1, \nu\, T_1] \quad (4.4)$$

So that

$$\int_{T_1}^{\nu T_1} e^{-\lambda t} M_X(t-\tau)\, d\tau = \int_{T_1}^{t-(\nu-1)\nu T_1} e^{-\lambda\tau} M_X(t-\tau) d\tau$$

$$= \int_{t-(\nu-1)\nu T_1}^{t-(\nu-2)\nu T_1} e^{-\lambda\tau} M_X(t-\tau)\, d\tau$$

$$+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$+ \int_{t-T_1}^{t-T_1} e^{-\lambda\tau} M_X(t-\tau)\, d\tau$$

$$+ \int_{t-T_1}^{\nu T_1} e^{-\lambda\tau} M_X(t-\tau)\, d\tau \quad (4.5)$$

But, we have

$$\int_{T_1}^{t-(\nu-1)T_1} e^{-\lambda\tau} M_X(t-\tau) d\tau = \int_{T_1}^{t-(\nu-1)\,T_1} e^{-\lambda\tau} e^{-\lambda(t-\tau)} \sum_{j=0}^{\upsilon-1} \frac{(\lambda m)^j}{j!} (t-\tau-jT_1)^j d\tau$$

$$= e^{-\lambda\tau}\sum_{j=0}^{\upsilon-1}\frac{(\lambda m)^j}{(j+1)!}[(t-(j+1)T_1)^{j+1} - ((\nu-1-j)T_1)^{j+1}];$$

$$\int_{t-(\nu-1)T_1}^{t-(\nu-2)T_1} e^{-\lambda\tau} M_X(t-\tau)\,d\tau = \int_{t-(\nu-1)T_1}^{t-(\nu-2)T_1} e^{-\lambda\tau} e^{-\lambda(t-\tau)} \sum_{j=0}^{\upsilon-2}\frac{(\lambda m)^j}{j!}(t-\tau-jT_1)^j d\tau$$

$$= e^{-\lambda\tau}\sum_{j=0}^{\upsilon-2}\frac{(\lambda m)^j}{(j+1)!}\left[\left((\nu-1-j)T_1\right)^{j+1} - ((\nu-2-j)T_1)^{j+1}\right];$$

$$\int_{t-2T_1}^{t-T_1} e^{-\lambda\tau} M_X(t-\tau)\,d\tau = \int_{t-2T_1}^{t-T_1} e^{-\lambda\tau} e^{-\lambda(t-\tau)} \sum_{j=0}^{1}\frac{(\lambda m)^j}{j!}(t-\tau-jT_1)^j d\tau$$

$$= e^{-\lambda\tau} e^{-\lambda(t-\tau)} \sum_{j=0}^{1}\frac{(\lambda m)^j}{(j+1)!}[((2-j)T_1)^{j+1} - ((1-j)T_1)^{j+1}]$$

$$\int_{t-T_1}^{\nu T_1} e^{-\lambda\tau} M_X(t-\tau)\,d\tau = \int_{t-T_1}^{\nu T_1} e^{-\lambda\tau} e^{-\lambda(t-\tau)}\,d\tau = e^{-\lambda\tau}\{(\nu+1)T_1 - t\}$$

And hence on addition of the above equations and submitting in (4.5), we get

$$M_X(t) = e^{-\lambda\tau}\left\{\sum_{j=0}^{\nu}\frac{(\lambda m)^j}{j!}(t-jT_1)^j - \lambda m(t-\nu T_1)\right\}, \nu T_1 < t \le (\upsilon+1)T_1 \qquad (4.6)$$

Sub case: $l = \nu + 1$

Here, we clearly have $(\nu+1)T_1 < t \le (\upsilon+2)T_1$ so that $(\nu-1)T_1 < t - 2T_1 \le \nu T_1$. We split the interval $(T_1, \nu T_1]$ into subintervals.

$$(T_1, t-\nu T_1], (t-\nu T_1, t-(\nu-1)T_1], \ldots\ldots (t-3T_1, t-2T_1, \nu T_1]$$

And write

$$\int_{T_1}^{\nu T_1} e^{-\lambda\tau} M_X(t-\tau)\,d\tau = \int_{T_1}^{t-\nu T_1} e^{-\lambda\tau} M_X(t-\tau)d\tau + \int_{t-\nu T_1}^{t-(\nu-1)T_1} e^{-\lambda\tau} M_X(t-\tau)d\tau + \cdots\ldots$$

$$+ \int_{t-3T_1}^{t-2T_1} e^{-\lambda\tau} M_X(t-\tau)d\tau + \int_{t-2T_1}^{\nu T_1} e^{-\lambda\tau} M_X(t-\tau)d\tau \qquad (4.7)$$

Using (4.4) and (4.6) in (4.7) and then substituting in (4.5), we get

$$M_X(t) = e^{-\lambda\tau}\left\{\sum_{k=0}^{[\nu+1]}\frac{(\lambda m)^j}{j!}(t-jT_1)^j - \lambda m(t-\nu T_1) - (\lambda m)^2(t-(\nu+1)T_1)^2\right\}$$

Where $(\nu+1)T_1 < t \le (\upsilon+2)T_1$. Proceeding inductively, we obtain

$$M_X(t) = e^{-\lambda\tau}\sum_{k=0}^{[t/_v]}(-1)^k\frac{(\lambda m)^k}{k!}\sum_{j=0}^{l-vt}\frac{(\lambda m)^j}{j!}(t-(j+\nu k)T_1)^{j+k} \qquad (4.8)$$

Where $(\nu+k)T_1 < t \le (\nu+k+1)T_1, k = 0,1,2,\ldots$ The equations (4.2) , (4.4) and (4.8) can be put together and we have

$$M_X(t) = e^{-\lambda t}\sum_{k=0}^{[l/_v]}(-1)^k\frac{(\lambda m)^k}{k!}\sum_{j=0}^{l-\nu k}(t-(j+\nu k)T_1)^{j+k} \qquad (4.9)$$

Where $lT_1 < t \le (l+1)T_1, l = 0,1,2\ldots.$ Next, we proceed to obtain $M_Y(t)$. Using the path structure of $\mathrm{X}(t)$. We have

$$M_Y(t) = \int_0^t M_X(\tau)\,d\tau \qquad (4.10)$$

When $0 < t \le T_1$, we have $M_X(t) = e^{-\lambda t}$ and so by (4.10), we get

$$M_Y(t) = \frac{1-e^{-\lambda t}}{\lambda}$$

When $lT_1 < t \le (l+1)T_1, l = 0,1,2 ....$, we write (2.10) ion the following manner

$$M_Y(t) = \sum_{n=1}^{l} \int_{(n-1)T_1}^{nT_1} M_X(\tau)d\tau + \int_{lT_1}^{t} M_X(\tau)d\tau \qquad (4.11)$$

Substituting (4.9) into (4.11), we obtain

$$M_Y(t) = \sum_{n=1}^{l} \sum_{k=0}^{[(n-1)/v]} (-1)^k \frac{(\lambda m)^k}{k!} \left\{ \sum_{j=0}^{(n-1-vk)} \frac{(\lambda m)^j}{j!} U\big((n-1)T_1, (j+vk)T_1, j+k\big) - U(nT_1, (j+vk)T_1, j+k) \right\}$$
$$+ \sum_{k=0}^{[(n-1)/v]} (-1)^k \frac{(\lambda m)^k}{k!} \sum_{j=0}^{(l-vk)} \frac{(\lambda m)^j}{j!} \{U(lT_1, (j+vk)T_1, j+k) - U(t, (j+vk)T_1, j + k\} \qquad (4.12)$$

Where

$$U(s,\theta,n) = \frac{e^{-\lambda s} n!}{\lambda^{n+1}} \left\{ 1 + \sum_{i=1}^{n} \frac{(\lambda(s-\theta))^i}{i!} \right\}$$

let $v \to \infty$ in (4.9), we get

$$M_X(t) = e^{-\lambda t} \sum_{j=0}^{t} \frac{(\lambda m)^j}{j!} (t - jT_1)^j$$

A result of parthasarathy (1979). Likewise, let $v \to \infty$ in(4.12), we obtain

$$M_Y(t) = \sum_{j=0}^{t} \frac{m^j}{\lambda} \left\{ e^{-\lambda jT_1} - e^{-\lambda t} \sum_{i=0}^{j} \frac{\big(\lambda(t - jT_1)\big)^i}{i!} \right\} (t - jT_1)^j, lT_1 < t \le (l+1)T_1, \quad l = 1,2 ...$$

a result of Udayabaskaran and Sudalaiyandi (1986).

V. THE LIMIT DISTRIBUTION OF $Y(t)$

We assume the $\lambda \le \mu + \beta(1-\delta)$. Then, as $t \to \infty$, $M_X(t) \to 0$, and so $X(t) \to 0$ almost surely. Consequently, the integral $\int_0^\infty X(\tau)\,d\tau$ exists almost surely and $Y(t)$ converges to the random variable $Y = \int_0^\infty X(\tau)d\tau$. In order to obtain explicit expression for the probability density function of Y, we assume further that

$$Re(r_2) < 1 < Re(r_1),$$

we have

$$\phi(s) = \frac{1}{2\lambda}\left\{ s + \lambda + \mu + \beta(1-\delta) - \sqrt{(s+\lambda+\mu+\beta(1-\delta))^2 - 4\lambda(\mu+\beta(1-\delta))} \right\} \qquad (5.1)$$

Denoting the p.d.f of Y by $f(y)$, we note that,

$$\phi(s) = \int_0^\infty e^{-sy} f(y)dy$$

that is, $\phi(s)$is the laplace transform of $f(y)$, and so inversion of $\phi(s)$ yields $f(y)$. To do the inversion, we expand (5.1) as we given below

$$\phi(s) = \frac{1}{2\lambda}\left\{ s + \lambda + \theta - (s+\lambda+\theta)\sqrt{1 - \frac{4\lambda\theta}{(s+\lambda+\theta)^2}} \right\} \qquad (5.2)$$

Where $\theta = \mu + \beta(1-\delta)$. Using Taylor's expansion in (5.2), we have

$$\phi(s) = \frac{1}{\lambda} \sum_{r=0}^{\infty} \frac{(2r)!}{r!\,(r+1)!} \frac{(\lambda\theta)^{r+1}}{(s+\lambda+\theta)^{2r+1}} \qquad (5.3)$$

Now inverting (5.3), we have explicitly

$$f(y) = (\mu + \beta(1-\delta))e^{-(\lambda+\mu+\beta(1-\delta))y} \sum_{r=0}^{\infty} \frac{(\lambda(\mu+\beta(1-\delta)y^2)^r}{r!\,(r+1)!}$$

Which may be equalently expressed in terms of l_1 bessel function as

$$f(y) = \theta e^{-(\lambda+\theta)y} \frac{1}{y\sqrt{\lambda\theta}} l_1\{2y\sqrt{\lambda\theta}\}(0 \le y < \infty)$$

The first two moments of Y are given by

$$E(Y) = -\frac{d}{ds}\{\phi(s)\}_{s=0} = \frac{1}{\mu + \beta(1-\delta) - \lambda} \qquad (5.4)$$

$$E(Y^2) = -\frac{d^2}{ds^2}\{\phi(s)\}_{s=0} = \frac{2(\mu + \beta(1-\delta))}{(\mu + \beta(1-\delta) - \lambda)^3}$$

and hence the variance of Y is given by

$$Var(Y) = \frac{\lambda + \mu + \beta(1-\delta))}{(\mu + \beta(1-\delta) - \lambda)^3} \qquad (5.5)$$

Setting $\beta = 0$ in (5.4) and (5.5) ,
we recover the results of Puri(1966)

$$E(Y) = \frac{1}{\mu - \lambda}$$

$$Var(Y) = \frac{\mu + \lambda}{(\mu - \lambda)^3}$$

VI.CONCLUSION

In this paper, it was shown that if $T_1 = 0$ and $T_2 = \infty$, the branching process $X(t)$ becomes age dependent can be described by Markov-modulated branching processes with p.g.f . when $T_1 < t < T_2$, then $X(t)$ becomes the modified Markov branching processes. We can branch out to when $F(t) = 1 - e^{-\lambda t}$ and $X(t)$ as the restricted branching process. Finally, we note that other statistically relevant quantities such as expectations of particle restricted moments can be computed using similar generating function techniques applies analogously when sparsity is present.

REFERENCES

[1] Assmussen, S. and Hering, H. (1983): Branching Processes. Birkhauer, Boston.

[2] Athreya, K.B. and Karlin, S. (1971a): Branching processes with random environments. Bull. Amer. math. Soc. 76, 865-870.

[3] Bellman, R. and Harris, T.E. (1948); On the theory of age-dependent stochastic branching processes. proc. Nat. Acad. Sci., New York., 34, 601-604.

[4] Finkelstien, M., Tucker, H.G. and Veeh, J.A. (1990): The limit distribution of the number of rare mutants. Journal Application probability. 2, 225-268.

[5] Harrington, D.P. and Fleming, T.R. (1978): Estimation for branching processes with Varying and random environments, math, Biosci., 39, 255-271.

[6] Ito, K. (1944): Stochatic Integrals, Proc. Imp. Acad., 20, 519-524.

[7] Parthasarathy, R. and Krishnamoorthy, A. (1979): Markov branching processes with varying growth rates. bull. math. boil. 41, 607-610.

[8] Parthasarathy, R. (1979): on a Modified markov branching processes. J. Math. Biol. 7, 95-97.

[9] Puri, P.S. (1969a): Some limit theorems on the branching processes and certain related processes, Sankhya, A31, 57-74.

[10] Srinivasan, S.K. and Udayabaskaran, S. (1982): A generalized branching process and a stochastic Integral associated with it, Journal of math. Phys. Sci., 16, 367-382.

[11] Thang, D.H. (1995): Some aspects of the theory of Stochastic Integrals. Vietnam J. of Math. 23, 1-28.

[12] Udayabaskaran, S. and Sudalaiyandi, G. (1986): On a Stochastic Integral of a branching process. J. Math. Biol. 24, J. Math. Biol. 24. 467- 472.